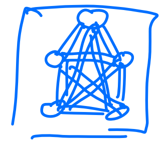
Lecture 3

$ex(n, H) = \max\{ e(G) : |G| = n,\ H \not\subseteq G \}$ $\quad |V(G)| = n$



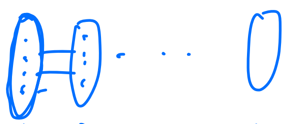
Turán theorem $ex(n, K_{r+1}) \leq (1 - \frac{1}{r}) \times \frac{n^2}{2}$

H: $\quad ex(n, H) = \qquad \chi(H)$:

Def. The chromatic number $\chi(H)$ of a graph H is the smallest number c such that the vertices of H can be colored with c colors and no two vertices of the same color are adjacent.

$V(G): V_1 \cup V_2 \cup \cdots V_k \qquad V_i$: independent set

1 2 k



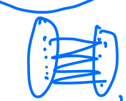
1 2 V_i

edge coloring

total coloring

defective coloring

$\chi(H) \leq 2 \iff H$ is bipartite graph $\iff H$ contains no odd cycle.



1 2

$V_1 \quad V_2$

Theorem 1 (Erdős–Stone–Simonovits) $\forall$ fixed graph H, $\forall \varepsilon > 0$, $\exists n_0$ s.t. $\forall n \geq n_0$

$$\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right) \times \frac{n^2}{2} \leq ex(n, H) \leq \left(1 - \frac{1}{\chi(H)-1} + \varepsilon\right) \times \frac{n^2}{2}$$

$\boxed{1 - \frac{1}{\chi(H)-1}}$ $\quad H: K_{r+1} \quad \chi(H) = r+1 \quad (1 - \frac{1}{r+1-1}) = (1 - \frac{1}{r})$

$\boxed{\chi(H) = 2}$: degenerate Turán problems $\quad (1 - \frac{1}{2-1} + \varepsilon) \times \frac{n^2}{2} = \varepsilon n^2$

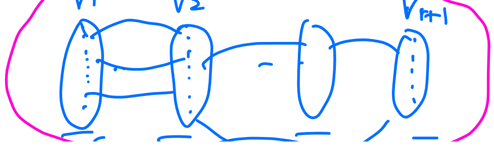
C_{2k} open problem $\quad C_8$

proof. $ex(n, H) \leq (1 - \frac{1}{\chi(H)-1} + \varepsilon) \times \frac{n^2}{2}$.

$G: |V(G)| = n \quad e(G) > (1 - \frac{1}{\chi(H)-1} + \varepsilon) \times \frac{n^2}{2}$

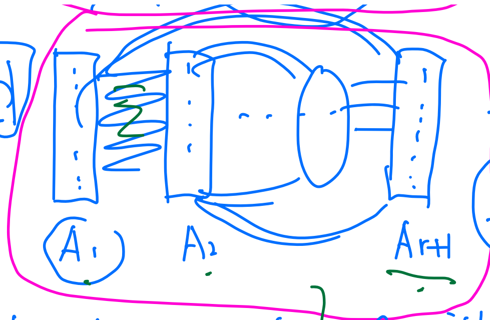
$\Rightarrow H \subset G. \quad H: \triangle, K_4, K_{r+1} \quad H:$

$\chi(H)$: $r+1$



$V_1 \quad V_2 \quad V_{r+1}$

$|A_i| \geq t \geq |V(H)|$



$\rightarrow$ Complete $(r+1)$-partite graph.

$\subseteq G$

$|A_i| \geq t \geq |V(H)|$

Lemma 1: $\forall\, r,\ t$ and any positive ε with $(\varepsilon < 1/r)$, $\exists\, n_0$ such that the following holds. Any graph G with $n \geq n_0$ vertices and $e > (1-\frac{1}{r}+\varepsilon)\times\frac{n^2}{2}$ edges contains $(r+1)$ disjoint set of vertices $A_1, A_2, \cdots A_{r+1}$ of size t such that the graph between A_i and A_j are complete, for every $1 \leq i < j \leq r+1$.

"cleaning" $\quad G \quad e(G) > (1-\frac{1}{r}+\varepsilon)\times\frac{n^2}{2} \longrightarrow G' \qquad \delta(G') \geq \cdots\cdots$

$\boxed{\delta(G') \geq (1-\frac{1}{r}+\frac{\varepsilon}{2})|V(G')|}$

proof. ① We find a subgraph G' of G such that
$\delta(G') \geq (1-\frac{1}{r}+\varepsilon/2)\,|V(G')|$.

$\delta(G) \geq (1-\frac{1}{r}+\varepsilon/2)|V(G)| \quad \checkmark \qquad G' = G.$

$\delta(G) < (1-\frac{1}{r}+\varepsilon/2)|V(G)| \quad \exists\ v_1 \quad d(v_1) < (1-\frac{1}{r}+\varepsilon/2)|V(G)|.$

delete v_1 $\quad G\backslash v_1 = G_1$ if $\delta(G_1) \geq (1-\frac{1}{r}+\varepsilon/2)|V(G_1)| \ \checkmark \quad G' = G_1$

Otherwise $\exists\ v_2 \quad d_{G_1}(v_2) < (1-\frac{1}{r}+\varepsilon/2)(V(G_1))$ delete v_2 in G_1

$G_2 = G_1 \backslash v_2 \quad \cdots\cdots\cdots \qquad |V(G')| = n' \qquad n' = ?$

We count the total number of edges removed.

$$\leq (1-\frac{1}{r}+\varepsilon/2)\times n + (1-\frac{1}{r}+\varepsilon/2)\times(n-1) + (1-\frac{1}{r}+\varepsilon/2)\times(n-2) + \cdots + (1-\frac{1}{r}+\varepsilon/2)(n'+1)$$

$$= \sum_{\ell=n'+1}^{n}(1-\frac{1}{r}+\frac{\varepsilon}{2})\cdot\ell = (1-\frac{1}{r}+\frac{\varepsilon}{2})\sum_{\ell=n'+1}^{n}\ell = (1-\frac{1}{r}+\frac{\varepsilon}{2})\times\frac{(n+n'+1)(n-n')}{2}$$

$$< (1-\frac{1}{r}+\varepsilon/2)\frac{n^2-n'^2}{2} + \frac{n-n'}{2}$$

$e(G') \leq \frac{n'^2}{2}$

G'

$\delta(G') \geq (1-\frac{1}{r}+\varepsilon/2)|V(G')|$

removed $\qquad$ remaining

$$e(G) \leq \boxed{(1-\frac{1}{r}+\varepsilon/2)\frac{n^2-n'^2}{2} + \frac{n-n'}{2}} + \frac{n'^2}{2}$$

$$= (1-\frac{1}{r}+\varepsilon/2)\times\frac{n^2}{2} + (\frac{1}{r}-\frac{\varepsilon}{2})\times\frac{n'^2}{2} + \frac{n-n'}{2}$$

$-(1-\frac{1}{r}+\varepsilon/2)+1$

$= \frac{1}{r}-\varepsilon/2$

$e(G) > (1-\frac{1}{r}+\varepsilon)\cdot\frac{n^2}{2}$

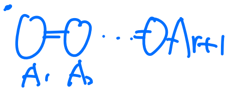
$$(1-\frac{1}{r}+\cancel{\frac{\varepsilon}{2}})\times\frac{n^2}{2} + (\frac{1}{r}-\frac{\varepsilon}{2})\times\frac{n'^2}{2} + \frac{n-n'}{2} < (1-\frac{1}{r}+\varepsilon)\times\frac{n^2}{2}$$

$$(\frac{1}{r}-\frac{\varepsilon}{2})\times\frac{n'^2}{2} - \frac{n}{2} < \frac{\varepsilon n^2}{4} - \frac{n}{2}$$

Therefore the process stops once $\boxed{n' \approx \sqrt{\varepsilon r}\, n}$

We will assume that we are working with this "good" subgraph G'.

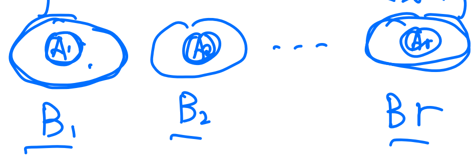
$n' = v(G') \approx \sqrt{\varepsilon r}\, n \qquad \delta(G') \geq (1-\frac{1}{r}+\frac{\varepsilon}{2})\cdot|V(G')|$.

We will show, by induction on r, $\quad G' \supseteq$ 

that there are $(r+1)$ sets $A_1, A_2 \cdots A_{r+1}$ of size t $\qquad |A_i| \geq t$

such that every edge between A_i and A_j is in G' with $1 \leq i < j \leq r+1$.

$\boxed{r=0}$ 

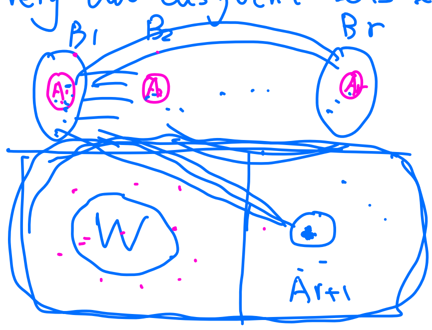


Given $r>0$, and $\lceil 3t/\varepsilon \rceil = s$.

We apply the induction hypothesis to find r disjoint sets $B_1, B_2, \cdots, B_r$ of size s such that the graph between every two disjoint sets is complete.

Let $U = V(G') \setminus \{B_1 \cup B_2 \cdots \cup B_r\}$

Let W be the set of vertices in U which are adjacent to at least t vertices in each B_i. $\quad |W| \to$ "large"



$|B_i| \geq s$

$|A_i| \geq t$

U

We will count the number of "missing" edges between U and $B_1 \cup \cdots \cup B_r$. $\quad$ candidate

$$\hat{m} \geq |U \setminus W| \times (s-t) \geq (n' - rs - |W|) \times (s - \frac{\varepsilon}{3}s)$$

$$\geq (n' - rs - |W|) \times (1-\frac{\varepsilon}{3})s$$

Every vertex in G' has at least $(1-\frac{1}{r}+\frac{\varepsilon}{2})n'$ neighbors in G'.

$(\delta(G') \geq (1-\frac{1}{r}+\frac{\varepsilon}{2})\cdot n')$

Every vertex in G' has at most $n' - (1-\frac{1}{r}+\frac{\varepsilon}{2})\cdot n' = (\frac{1}{r}-\frac{\varepsilon}{2})n'$ missing edges.

$\hat{m} \leq (r\cdot s)\times(\frac{1}{r}-\frac{\varepsilon}{2})n' = (1-\frac{r\varepsilon}{2})\cdot s n'$ $\quad$ "double counting"

$$(n' - rs - |W|) \times \left(1 - \frac{\varepsilon}{3}\right) s \le rs \times \left(\frac{1}{r} - \frac{\varepsilon}{2}\right) n' = \left(1 - \frac{r\varepsilon}{2}\right) s n'$$

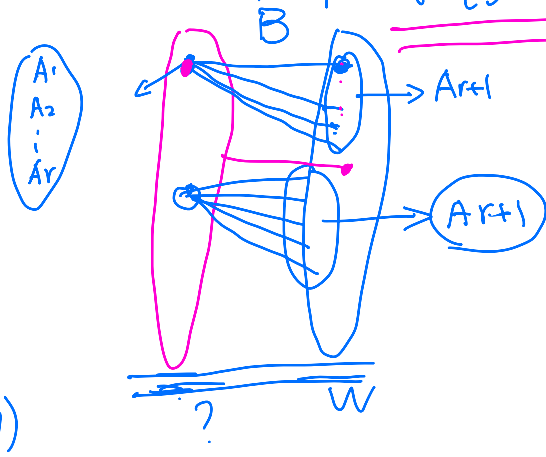
$$|W| \times \left(1 - \frac{\varepsilon}{3}\right) \times s \geq (n' - rs)\left(1 - \frac{\varepsilon}{3}\right) s - \left(1 - \frac{r\varepsilon}{2}\right) \cdot s n'$$

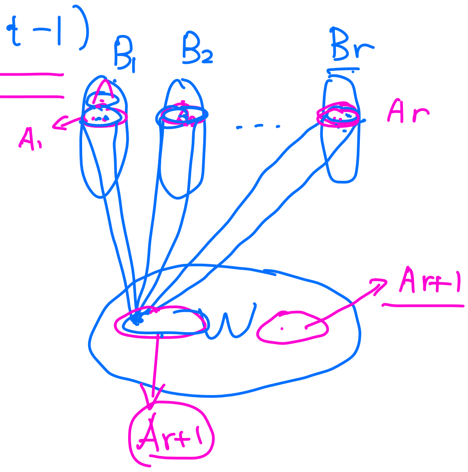
$$= \varepsilon\left(\frac{r}{2} - \frac{1}{3}\right) s n' - \left(1 - \frac{\varepsilon}{3}\right) r s^2$$

$|W| \to$ large

Since ε, r and s are constants, we can make $|W|$ large by making n' large.

$$|W| > \binom{s}{t}^r \cdot (t-1)$$





$$e(B) \le \boxed{\binom{s}{t}^r} \times (t-1)$$

$$e(B) \geq |W| \times 1 > \binom{s}{t}^r (t-1)$$

$\boxed{t = ?}$ $t = f(n)$?

proof of Erdős–Stone–Simonovits.

For the upper bound, if $\chi(H) = r+1$, then H can be embedded in a graph G consisting of $r+1$ sets $A_1, A_2, \ldots, A_{r+1}$ of size t such that the graph between any two disjoint A_i and A_j is complete, provided t is large enough.

For the lower bound, we consider the Turán graph given by $r = \chi(H) - 1$ sets of almost equal size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

This has roughly the required number of vertices and it is clear that every subgraph of this graph has chromatic number at most r $(\chi(H) - 1)$.

$\boxed{\chi(H) = 2}$

Lecture 4. Regularity lemma.

Szemerédi. $\forall \delta > 0$, $\forall k \geq 3$, $\exists n_0$ such that for $n \geq n_0$ any subset of $\{1, 2, \cdots, n\}$ with at least δn elements contains an arithmetic progression of length k.

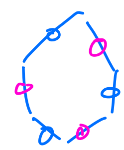
$x, \quad x+d, \quad x+2d, \quad x+3d, \quad \cdots$

$x + y = z$

$x + y = 2z$

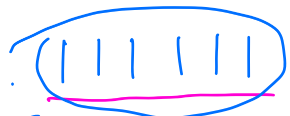
(Abel prize, 2012)

: for his fundamental contribution to discrete mathematics and theoretical computer science, and in recognition of the profound and lasting impact of these contribution on additive number theory and ergodic theory"



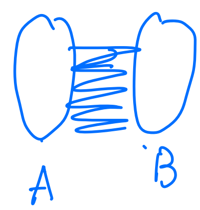
Hajal - Szemerdi Theorem

$\delta(G) \geq n/2 \rightarrow$ Hamilton cycle



$\delta(G) \geq \frac{2}{3} n \rightarrow \Delta$- factor

$\delta(G) \geq (1 - \frac{1}{r}) n \Rightarrow K_r$-factor



Let G be a graph.

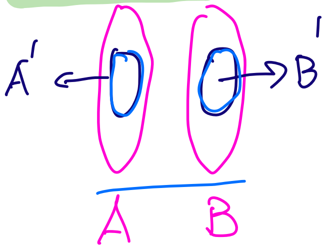
$E(A, B)$ $\quad d(A, B) = \dfrac{|E(A,B)|}{|A||B|}$.

Def 1. Let G be a graph, and let A and B be two subsets of the vertex set. The pair (A, B) is "ε-regular" if for every $A' \subset A$ and $B' \subset B$

$d(A, B) = 0$

$d(A, B) = 1$

with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, $\quad |d(A', B') - d(A, B)| \leq \varepsilon$.



$d(A, B)$

A partition $V(G) = X_1 \cup X_2 \cdots \cup X_k$ is ε-regular if

$$\sum \frac{|X_i||X_j|}{n^2} \leq \varepsilon,$$

where the sum is taken over all pairs (X_i, X_j) which are not ε-regular.

Theorem 1. (Szemerédi regularity lemma)

$\forall \varepsilon > 0$, $\exists M$ $(= M(\varepsilon))$ such that for every graph G, there is an ε-regular partition of the vertex set of G with at most M pieces.

① proof of regularity lemma.

② use it to prove $\begin{cases} \text{Erdős–Stone–Simonovits theorem.} \\ k\text{-AP.} \end{cases}$